

# Conformal Blocks for Arbitrary Spins in Two Dimensions

H. Osborn<sup>1</sup>

Department of Applied Mathematics and Theoretical Physics,  
Wilberforce Road, Cambridge, CB3 0WA, England

## Abstract

Conformal blocks for the finite dimension conformal group  $SO(2, 2)$  for four point functions for fields with arbitrary spins in two dimensions are obtained by evaluating an appropriate integral. The results are just products of hypergeometric functions of the conformally invariant cross ratios formed from the four complex coordinates. Results for scalars previously obtained are a special case. Applications to four point functions involving the energy momentum tensor are discussed.

---

<sup>1</sup>ho@damtp.cam.ac.uk

There has been a resurgence of interest in applying bootstrap methods to understanding the structure of conformal and superconformal field theories. Bounds on the anomalous dimensions depending on general principles such as conformal symmetry the operator product expansion and crossing have been analysed. Most applications have been to four dimensional theories [1, 2] but very recently [3] the three dimensional Ising model has been investigated with remarkable indications that this may become a very effective method for determining critical exponents.

In order to apply the bootstrap equations it is necessary to be able to determine the conformal blocks representing the contributions of particular conformal primary operators and their descendants to a four point function for conformal primary operators when the operator product expansion is applied to distinct pairs of operators. Equating the presumed convergent sums over conformal blocks in different channels at convenient points leads, together with positivity conditions arising from unitarity, to non trivial constraints. In general conformal blocks depend on two conformal invariants  $u, v$  and are the extensions to the non compact conformal group  $SO(d, 2)$ , or  $SO(d + 1, 1)$ , of two variable harmonic polynomials for the corresponding compact  $SO(d + 2)$ . In two and four dimensions simple expressions for the conformal blocks for four point functions for scalar operators were found in terms of hypergeometric functions of variables  $z, \bar{z}$  which are simply defined in terms of  $u, v$  [4]. Recently these results have been extended to include cases when the four point function involves fields with spin [5, 6, 7] although concise simple generalisations of the spinless case for arbitrary spins have not so far been found.

In this note we show how conformal blocks for four point functions for arbitrary spins may be found in two dimensions. The results are just a simple extension of the spinless case. Of course two dimensions is very special, not least because the associated rotational group  $SO(2)$  has only one dimensional representations labelled by spin or helicity  $s$ . The difficulties of handling multi-index tensors or spinors are part of the complexities that arise in the four dimensional case but progress has been made in handling these [5, 7] using twistor methods.

Our starting point is an integral representation which immediately gives rise to a conformal block. The conformally covariant integrals which are relevant in two dimensions have the form, written in terms of standard complex variables  $z, \bar{z}$  with  $d^2z = d\text{Re}z, d\text{Im}z$ ,

$$I_n = \frac{1}{\pi} \int d^2z \prod_{i=1}^n \frac{1}{(z - z_i)^{\alpha_i}} \frac{1}{(\bar{z} - \bar{z}_i)^{\bar{\alpha}_i}}, \quad \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \bar{\alpha}_i = 2, \quad \alpha_i - \bar{\alpha}_i \in \mathbb{Z}. \quad (1)$$

This was evaluated in general in [8] for  $n = 2, 3, 4$ .

In two dimensions the conformal primary fields  $\varphi(z, \bar{z})$  are labelled by  $h, \bar{h}$  with the scale dimension  $\Delta = h + \bar{h}$  and spin  $s = h - \bar{h}$  [9] and we require  $2s \in \mathbb{Z}$ , with  $\varphi$  bosonic/fermionic according to whether  $2s$  is even/odd. For any  $\varphi(z, \bar{z})$  conjugation gives a conformal primary field  $\bar{\varphi}(z, \bar{z})$  with  $h \leftrightarrow \bar{h}$ . For real scalars  $h = \bar{h}$  we may impose  $\varphi(z, \bar{z}) = \varphi(\bar{z}, z)$ . Corresponding to  $\varphi(z, \bar{z})$  we define a dual or shadow field

$$\tilde{\varphi}(z, \bar{z}) = K_{h, \bar{h}} \frac{1}{\pi} \int d^2y \frac{1}{(z - y)^{2-2h} (\bar{z} - \bar{y})^{2-2\bar{h}}} \varphi(y, \bar{y}), \quad (2)$$

which is a conformal primary with  $\tilde{h} = 1 - h, \bar{\tilde{h}} = 1 - \bar{h}$ . Choosing

$$K_{h,\bar{h}} = \frac{\Gamma(2-2\bar{h})}{\Gamma(2h-1)} = (-1)^{2(h-\bar{h})} \frac{\Gamma(2-2h)}{\Gamma(2\bar{h}-1)}, \quad (3)$$

ensures, using (1) for  $n = 2$ , that  $\tilde{\varphi} = (-1)^{2(h-\bar{h})}\varphi$ .

The operator product expansion for  $\varphi_1\varphi_2$  contains all operators  $\mathcal{O}$  for which there is a non zero three point function. For  $\varphi_1, \varphi_2, \mathcal{O}$  conformal primaries<sup>1</sup> this is determined by conformal symmetry so that

$$\begin{aligned} \langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \mathcal{O}(z, \bar{z}) \rangle &= (-1)^{2(h-\bar{h})} \langle \mathcal{O}(z, \bar{z}) \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \rangle \\ &= C_{12\mathcal{O}} \mathcal{F}_{12}^{h\bar{h}}(z, \bar{z}), \end{aligned} \quad (4)$$

where we require  $h_1 - \bar{h}_1 + h_2 - \bar{h}_2 + h - \bar{h} \in \mathbb{Z}$  and

$$\begin{aligned} \mathcal{F}_{12}^{h\bar{h}}(z, \bar{z}) &= \frac{1}{z_{12}^{h_1+h_2-h} (z_1 - z)^{h+h_{12}} (z_2 - z)^{h-h_{12}}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}} (\bar{z}_1 - \bar{z})^{\bar{h}+\bar{h}_{12}} (\bar{z}_2 - \bar{z})^{\bar{h}-\bar{h}_{12}}} \\ &= (-1)^{2(h-\bar{h})} \frac{1}{z_{12}^{h_1+h_2-h} (z - z_1)^{h+h_{12}} (z - z_2)^{h-h_{12}}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}} (\bar{z} - \bar{z}_1)^{\bar{h}+\bar{h}_{12}} (\bar{z} - \bar{z}_2)^{\bar{h}-\bar{h}_{12}}}, \end{aligned} \quad (5)$$

defining  $z_{12} = z_1 - z_2$ ,  $h_{12} = h_1 - h_2$  and similarly for  $\bar{z}_{12}, \bar{h}_{12}$ . Using the result for  $I_3$  as defined in (1)

$$\begin{aligned} K_{h,\bar{h}} \frac{1}{\pi} \int d^2y \frac{1}{(z-y)^{2-2h} (\bar{z}-\bar{y})^{2-2\bar{h}}} \mathcal{F}_{12}^{h\bar{h}}(y, \bar{y}) \\ = \frac{\Gamma(1-h-h_{12}) \Gamma(1-\bar{h}+\bar{h}_{12})}{\Gamma(\bar{h}+\bar{h}_{12}) \Gamma(h-h_{12})} \mathcal{F}_{12}^{1-h, 1-\bar{h}}(z, \bar{z}). \end{aligned} \quad (6)$$

For the four point function  $\langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \varphi_3(z_3, \bar{z}_3) \varphi_4(z_4, \bar{z}_4) \rangle$ , requiring the spins to be constrained by  $\sum_{i=1}^4 (h_i - \bar{h}_i) \in \mathbb{Z}$ , the conformal block corresponding to the operator product expansion of  $\varphi_1\varphi_2$  and  $\varphi_3\varphi_4$  both containing the operator  $\mathcal{O}$  may be determined by evaluating the conformally covariant integral

$$\begin{aligned} &\frac{\Gamma(\bar{h}+\bar{h}_{12}) \Gamma(1-\bar{h}+\bar{h}_{34})}{\Gamma(1-h-h_{12}) \Gamma(h-h_{34})} \frac{1}{\pi} \int d^2z \mathcal{F}_{12}^{h\bar{h}}(z, \bar{z}) \mathcal{F}_{34}^{1-h, 1-\bar{h}}(z, \bar{z}) \\ &= (-1)^{2(h-\bar{h})} \frac{\Gamma(1-\bar{h}+\bar{h}_{12}) \Gamma(\bar{h}+\bar{h}_{34})}{\Gamma(h-h_{12}) \Gamma(1-h-h_{34})} \frac{1}{\pi} \int d^2z \mathcal{F}_{12}^{1-h, 1-\bar{h}}(z, \bar{z}) \mathcal{F}_{34}^{h\bar{h}}(z, \bar{z}) \\ &= \frac{1}{z_{12}^{h_1+h_2} z_{34}^{h_3+h_4}} \left( \frac{z_{24}}{z_{14}} \right)^{h_{12}} \left( \frac{z_{14}}{z_{13}} \right)^{h_{34}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2} \bar{z}_{34}^{\bar{h}_3+\bar{h}_4}} \left( \frac{\bar{z}_{24}}{\bar{z}_{14}} \right)^{\bar{h}_{12}} \left( \frac{\bar{z}_{14}}{\bar{z}_{13}} \right)^{\bar{h}_{34}} \mathcal{I}(\eta, \bar{\eta}), \end{aligned} \quad (7)$$

<sup>1</sup>Our use of the term conformal primary, requires just  $L_1\varphi_i(0,0) = \bar{L}_1\varphi_i(0,0) = 0$  as well as  $L_0\varphi_i(0,0) = h_i\varphi_i(0,0)$ ,  $\bar{L}_0\varphi_i(0,0) = \bar{h}_i\varphi_i(0,0)$  where  $L_{\pm}, L_0$  and  $\bar{L}_{\pm}, \bar{L}_0$ , with algebra  $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ , are the conformal generators. This use of the notion of conformal primary agrees with that in higher dimensions when it is necessary that  $\varphi(0)$  is annihilated by the conformal generator  $K_a$ . In two dimensional conformal field theories what is commonly referred to as a conformal primary field  $\varphi$ , where  $L_n\varphi_i(0,0) = \bar{L}_n\varphi_i(0,0) = 0$  for all  $n > 0$ , is here called a Virasoro primary.

for  $\eta, \bar{\eta}$  conformal invariants given here by

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad \bar{\eta} = \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}}. \quad (8)$$

The evaluation of  $I_4$  given in [8] leads to

$$\begin{aligned} \mathcal{I}(\eta, \bar{\eta}) &= \frac{(-1)^{2(h-\bar{h})}}{K_{1-h, 1-\bar{h}}} \frac{\Gamma(\bar{h} + \bar{h}_{12}) \Gamma(\bar{h} + \bar{h}_{34})}{\Gamma(1-h-h_{12}) \Gamma(1-h-h_{34})} F_{h\bar{h}}(\eta, \bar{\eta}) \\ &+ \frac{1}{K_{h, \bar{h}}} \frac{\Gamma(1-\bar{h} + \bar{h}_{12}) \Gamma(1-\bar{h} + \bar{h}_{34})}{\Gamma(h-h_{12}) \Gamma(h-h_{34})} F_{1-h, 1-\bar{h}}(\eta, \bar{\eta}), \end{aligned} \quad (9)$$

with

$$F_{h\bar{h}}(\eta, \bar{\eta}) = \eta^h F(h-h_{12}, h+h_{34}; 2h; \eta) \bar{\eta}^{\bar{h}} F(\bar{h}-\bar{h}_{12}, \bar{h}+\bar{h}_{34}; 2\bar{h}; \bar{\eta}). \quad (10)$$

The symmetry of (9) under  $h \rightarrow 1-h, \bar{h} \rightarrow 1-\bar{h}$ , up to a sign  $(-1)^{2(h-\bar{h})}$ , follows directly from (7) and reflects the contribution of both  $\mathcal{O}$  and its shadow. It is straightforward to separate  $F_{h\bar{h}}(\eta, \bar{\eta})$  as just the contribution corresponding to  $\mathcal{O}$  and thus (10) is just the conformal block for a conformal primary operator labelled by  $h, \bar{h}$ . A special case of (10) was actually given in [10], which gave the first detailed discussion of what are now termed conformal blocks. If  $h_{12} = h_{34} = \bar{h}_{12} = \bar{h}_{34} = 0$   $F_{00} = 1$  representing the identity.

The result (10) is in fact the leading, essentially trivial, term in the highly non trivial Virasoro conformal blocks, which contains contributions from all descendants generated by  $L_{-n}$  for  $n = 1, 2, \dots$ , as the central charge  $c \rightarrow \infty$  [11]. The Virasoro conformal blocks are of course expressible as a sum over contributions of the form (10) for all conformal primary descendants.

Conformal invariance for the four point function for  $\varphi_1 \varphi_2 \varphi_3 \varphi_4$  dictates

$$\begin{aligned} &\langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \varphi_3(z_3, \bar{z}_3) \varphi_4(z_4, \bar{z}_4) \rangle \\ &= \frac{1}{z_{12}^{h_1+h_2} z_{34}^{h_3+h_4}} \left( \frac{z_{24}}{z_{14}} \right)^{h_{12}} \left( \frac{z_{14}}{z_{13}} \right)^{h_{34}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2} \bar{z}_{34}^{\bar{h}_3+\bar{h}_4}} \left( \frac{\bar{z}_{24}}{\bar{z}_{14}} \right)^{\bar{h}_{12}} \left( \frac{\bar{z}_{14}}{\bar{z}_{13}} \right)^{\bar{h}_{34}} \mathcal{F}_{1234}(\eta, \bar{\eta}), \end{aligned} \quad (11)$$

and then the operator product expansion gives

$$\mathcal{F}_{1234}(\eta, \bar{\eta}) = \sum_{h, \bar{h}} a_{h\bar{h}} F_{h\bar{h}}(\eta, \bar{\eta}). \quad (12)$$

The sum in (12) includes for any  $\mathcal{O}$  labelled by  $h, \bar{h}$  also its conjugate  $\bar{\mathcal{O}}$  with contribution proportional to  $F_{\bar{h}h}(\eta, \bar{\eta})$ , with in general an independent coefficient. For  $\varphi_i$  real scalars,  $h_i = \bar{h}_i$  and  $F_{\bar{h}h}(\eta, \bar{\eta}) = F_{h\bar{h}}(\bar{\eta}, \eta)$  and  $a_{\bar{h}h} = a_{h\bar{h}}$  so that (11) and (12) can then be combined in the form

$$\begin{aligned} &\langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \varphi_3(z_3, \bar{z}_3) \varphi_4(z_4, \bar{z}_4) \rangle \\ &= \frac{1}{|z_{12}|^{2(h_1+h_2)} |z_{34}|^{2(h_3+h_4)}} \left( \frac{|z_{24}|}{|z_{14}|} \right)^{2h_{12}} \left( \frac{|z_{14}|}{|z_{13}|} \right)^{2h_{34}} \sum_{h \geq \bar{h}} a_{h\bar{h}} (F_{h\bar{h}}(\eta, \bar{\eta}) + F_{\bar{h}h}(\bar{\eta}, \eta)). \end{aligned} \quad (13)$$

This result is then in accord with the form of two dimensional conformal blocks obtained in [4] where symmetry under  $\eta \leftrightarrow \bar{\eta}$  played an essential role. Conformal blocks in two dimensions which were odd under  $\eta \leftrightarrow \bar{\eta}$  were considered in [12].

As a trivial application of (12) we may consider chiral fields  $\varphi, \bar{\varphi}$  such that

$$\langle \varphi(z_1) \varphi(z_2) \rangle = \frac{1}{z_{12}^{2h}}, \quad \langle \bar{\varphi}(\bar{z}_1) \bar{\varphi}(\bar{z}_2) \rangle = \frac{1}{\bar{z}_{12}^{2\bar{h}}}, \quad 2h, 2\bar{h} \in \mathbb{N}. \quad (14)$$

The four point functions involving  $\varphi, \varphi$  and  $\bar{\varphi}, \bar{\varphi}$  have only disconnected contributions determined by (14). They can be cast in the form (11) with

$$\mathcal{F}_{\varphi\varphi\bar{\varphi}\bar{\varphi}}(\eta, \bar{\eta}) = 1, \quad \mathcal{F}_{\varphi\bar{\varphi}\varphi\bar{\varphi}}(\eta, \bar{\eta}) = \eta^h \bar{\eta}^{\bar{h}}. \quad (15)$$

In the first case only the identity operator contributes in the operator product expansion, in the second since, with  $h_{12} \rightarrow h, \bar{h}_{34} \rightarrow -\bar{h}$ ,  $F_{h\bar{h}}(\eta, \bar{\eta}) = \eta^h \bar{\eta}^{\bar{h}}$ , corresponding to the conformal primary  $\varphi\bar{\varphi}$ .

It is interesting and less trivial to consider the role of the energy moment tensor  $T(z)$ , for which  $h = 2, \bar{h} = 0$ , and its conjugate  $\bar{T}(\bar{z})$ . In two dimensions if  $\varphi_i$  are Virasoro primary the four point correlation functions involving  $T(z)$ , and also  $\bar{T}(\bar{z})$ , with  $\varphi_1 \varphi_2 \varphi_3$  are fully determined by Ward identities [13]. Hence

$$\begin{aligned} & \langle T(z) \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \varphi_3(z_3, \bar{z}_3) \rangle \\ &= \sum_{i=1}^3 \left( \frac{h_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right) \langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \varphi_3(z_3, \bar{z}_3) \rangle, \end{aligned} \quad (16)$$

where, with the definition (5), we may take

$$\langle \varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) \varphi_3(z_3, \bar{z}_3) \rangle = C_{123} \mathcal{F}_{12}^{h_3 \bar{h}_3}(z_3, \bar{z}_3). \quad (17)$$

The result from (16) can then be cast in the form (11), with an appropriate relabelling, where

$$\begin{aligned} \mathcal{F}_{T123}(\eta, \bar{\eta}) &= C_{123} f(\eta) \bar{f}(\bar{\eta}), \quad \bar{f}(\bar{\eta}) = \bar{\eta}^{\bar{h}_1} (1 - \bar{\eta})^{-\bar{h}_1 - \bar{h}_{23}}, \\ f(\eta) &= \eta^{h_1} (1 - \eta)^{-h_1 - h_{23}} (h_1 (1 - \eta) + h_3 \eta - h_2 \eta (1 - \eta)). \end{aligned} \quad (18)$$

To obtain an expansion as in (12), where in (10) we now take  $h_{12} \rightarrow 2 - h_1, \bar{h}_{12} \rightarrow -\bar{h}_1$  and  $h_{34} \rightarrow h_{23}, \bar{h}_{34} \rightarrow \bar{h}_{23}$ , then since

$$\bar{\eta}^{\bar{h}_1} (1 - \bar{\eta})^{-\bar{h}_1 - \bar{h}_{23}} = \bar{\eta}^{\bar{h}_1} F(2\bar{h}_1, \bar{h}_1 + \bar{h}_{23}; 2\bar{h}_1; \bar{\eta}), \quad (19)$$

it is sufficient to consider just

$$f(\eta) = \sum_{n \geq 0} a_n \eta^{h_1 + n} F(2h_1 - 2 + n, h_1 + h_{23} + n; 2h_1 + 2n; \eta). \quad (20)$$

For  $n = 0, 1, 2$

$$a_0 = h_1, \quad a_1 = 0, \quad a_2 = \frac{h_1(h_1 - 1) - 3h_{23}^2 + (h_2 + h_3)(2h_1 + 1)}{2(2h_1 + 1)}. \quad (21)$$

The result for  $n = 0$ , reflecting the contribution of  $\varphi_1$  to the operator product expansion of  $T\varphi_1$ , is determined by Ward identities. If  $h_{23} = 0$ ,  $a_n = 0$  for all odd  $n$  and there is the general result

$$a_{2p} = \left( h_1 + h_2 \frac{2p}{h_1 - 1} (2h_1 - 1 + 2p) \right) \frac{(h_1 - 1)_{2p}}{(2h_1 - 2 + 2p)_{2p}}. \quad (22)$$

The results (19) and (20) are a consequence of the fact that the conformal primary operators  $\varphi_{1,n}$  present in the operator product expansion of  $T(z) \varphi_1(z_1, \bar{z}_1)$  have, for  $\varphi_1$  a Virasoro primary, just  $h = h_1 + n$ ,  $\bar{h} = \bar{h}_1$  with  $n = 0, 2, 3, \dots$ . The expansion has the form<sup>2</sup>

$$T(z) \varphi_1(0, 0) = \sum_{n=0, n \neq 1}^{\infty} z^{n-2} {}_1F_1(n+2; 2h_1+2n; z \partial) \varphi_{1,n}(0, 0), \quad (23)$$

for  $\varphi_{1,0} = h_1 \varphi_1$ , and where  $\partial^r \varphi_{1,n}(0, 0) \equiv \partial_y^r \varphi_{1,n}(y, 0)|_{y=0}$ . Since  $T(z) = \sum_n z^{-n-2} L_n$  (23) gives

$$L_{-n} \varphi_1 = \sum_{r=0, r \neq n-1}^n \binom{n+1}{r} \frac{1}{(2h_1 + 2n - 2r)_r} \partial^r \varphi_{1,n-r}, \quad n = 0, 1, 2, \dots, \quad (24)$$

which can be inverted for  $n = 2, 3, \dots$

$$\varphi_{1,n} = \sum_{r=0}^{n-2} \binom{n+1}{r} \frac{1}{(2h_1 + 2n - r - 1)_r} (-\partial)^r L_{-n+r} \varphi_1 - \frac{(n^2 - 1)(2h_1 + n)}{2(2h_1 + n - 1)_n} (-\partial)^n \varphi_1. \quad (25)$$

As is well known if  $\varphi_{1,n}$  is a Virasoro primary, so that  $L_2 \varphi_{1,n} = 0$ , there are further conditions relating  $h_1$  and the Virasoro central charge  $c$ .

Minimal models are characterised by vanishing of  $\varphi_{1,n}$  for some  $n$ , if  $\varphi_{1,2} = 0$  then in (21)  $a_2 = 0$ . For  $h_1 = h_2$  this requires  $h_3 = \frac{1}{3}(8h_1 + 1)$  which is satisfied by the Ising model which contains two Virasoro primaries  $\sigma$ ,  $h_\sigma = \bar{h}_\sigma = \frac{1}{16}$  and  $\epsilon$ ,  $h_\epsilon = \bar{h}_\epsilon = \frac{1}{2}$  and  $c = \frac{1}{2}$ . In minimal models correlation functions are then determined in terms of linear partial differential equations [13, 14], for four point functions these become ordinary differential equations in  $\eta$ . Such minimal models may serve as a testing ground for applications of conformal bootstrap methods in higher dimensions.

For any two dimensional conformal field theory the correlations functions of the energy momentum are universal depending only on  $c$ . For the four point function

$$\begin{aligned} \langle T(z_1) T(z_2) T(z_3) T(z_4) \rangle &= \frac{1}{z_{12}^4 z_{34}^4} \mathcal{F}_{TTTT}(\eta), \\ \mathcal{F}_{TTTT}(\eta) &= \frac{1}{4} c^2 (1 + \eta^4 + \eta^4 (1 - \eta)^{-4}) + 2c \eta^2 (1 - \eta)^{-2} (1 - \eta + \eta^2). \end{aligned} \quad (26)$$

---

<sup>2</sup>More generally the contribution of a conformal primary operator  $\mathcal{O}$  to the operator product of  $\varphi_1 \varphi_2$  is determined [4] by (4) and (5) to have the form

$$\varphi_1(z_1, \bar{z}_1) \varphi_2(z_2, \bar{z}_2) = \frac{C_{12\mathcal{O}}}{z_{12}^{h_1+h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2}} z_{12}^{h_1} F_1(h + h_{12}; 2h; z_{12} \partial_{z_2}) \bar{z}_{12}^{\bar{h}_1} F_1(\bar{h} + \bar{h}_{12}; 2\bar{h}; \bar{z}_{12} \partial_{\bar{z}_2}) \mathcal{O}(z_2, \bar{z}_2)$$

The conformal partial wave expansion then takes the form<sup>3</sup>

$$\begin{aligned}\mathcal{F}_{TTTT}(\eta) &= \frac{1}{4}c^2 + \sum_{p=0}^{\infty} a_{2p} \eta^{2p+2} F(2p+2, 2p+2; 4p+4; \eta), \\ a_{2p} &= \left( \frac{1}{144} c^2 (2p-1)_6 + 2c(1+2p(2p+3)) \right) \frac{(2p)!(2p+1)!}{(4p+1)!}.\end{aligned}\quad (27)$$

The conditions under which conformal primaries are absent from the operator product expansion of  $T(z)\varphi(0,0)$  can be obtained from the four point function  $\langle T\varphi T\varphi \rangle$  for which

$$\begin{aligned}\mathcal{F}_{T\varphi T\varphi}(\eta) &= \frac{1}{2}c\eta^{h+2} + \frac{\eta^h}{(1-\eta)^2}(h^2 - 2h\eta(1-\eta)) \\ &= \sum_{n=0}^{\infty} C_n \eta^{h+n} F(2h+n-2, n+2; 2h+2n; \eta).\end{aligned}\quad (28)$$

The expansion coefficients are given by

$$\begin{aligned}C_n &= \left( \frac{1}{12}(n-1)_3 (2h)_{n+1} c + 2h(n(n+2h-1)+1)(2h-2)_n \right) (-1)^n \frac{(2h-2)_n}{(2h-2)_{2n+1}} \\ &\quad + h((n+1)(2h-2+n)-2)(n+1)! \frac{(2h-2)_n}{(2h-2)_{2n+1}}.\end{aligned}\quad (29)$$

The condition  $C_n = 0$  for suitable  $c, h$  implies the absence of the conformal primary descendant  $\varphi_n$ .

In higher dimensions the four point function for the energy momentum tensor depends on the dynamical details of the particular theory except for  $\mathcal{N} = 4$  superconformal theories at strong coupling and also large  $N$  when it can be calculated for  $d = 4$  by using pure gravity on  $AdS_5$ . Although this would have significant interest present calculations [15] suffice only to determine the three point function. Results based on the  $AdS_4/CFT_3$  correspondence [16] give quite simple expressions for the four point function in three dimensions. The four point function for the energy momentum tensor is also the natural object to analyse in discussions of the existence of higher dimensional conformal field theories using bootstrap methods. To achieve this expressions for conformal blocks with external spin 2 are essential. The connections between conformal partial waves in even dimensions for external scalars suggest the possibility that this might be extended to non trivial external spins. If feasible the present results in two dimensions give a very simple starting point.

## Acknowledgements

I would like to thank João Penedones and Slava Rychkov for helpful remarks and also to Daniel Harlow for informing me of the extensive relevant literature partially listed in [11].

---

<sup>3</sup>The results can be obtained from

$$\eta^h = \sum_{p=0}^{\infty} (-1)^p \frac{(h)_p^2}{p!(2h+p-1)_p} \eta^{h+p} F(h+p, h+p; 2h+2p; \eta).$$

## References

- [1] R. Rattazzi, V.S. Rychkov, E. Tonni and A. Vichi, Bounding scalar operator dimensions in 4D CFT, JHEP 0812:031 (2008), arXiv:0807.0004 [hep-th];  
V.S. Rychkov and A. Vichi, Universal Constraints on Conformal Operator Dimensions, Phys. Rev. D80 (2009) 045006, 2009, arXiv:0905.2211 [hep-th];  
F. Caracciolo and V.S. Rychkov, Rigorous Limits on the Interaction Strength in Quantum Field Theory, Phys. Rev. D81 (2010) 085037, arXiv:0912.2726 [hep-th];  
R. Rattazzi, S. Rychkov and A. Vichi, Bounds in 4D Conformal Field Theories with Global Symmetry, J. Phys. A44 (2011) 35402, arXiv:1009.5985 [hep-th];  
R. Rattazzi, V.S. Rychkov and A. Vichi, Central Charge Bounds in 4D Quantum Field Theory, Phys. Rev. D83 (2011) 046011, arXiv:1009.2725 [hep-th];  
A. Vichi, Improved bounds for CFT's with global symmetries, JHEP 1201:162 (2012), arXiv:1106.4037 [hep-th];  
D. Poland, D. Simmons-Duffin and A. Vichi, Carving Out the Space of 4D CFTs, arXiv:1109.5176 [hep-th].
- [2] D. Poland and D. Simmons-Duffin, Bounds on 4D Conformal and Superconformal Field Theories, JHEP (2011) 1105:017, arXiv:1009.2087 [hep-th].
- [3] Sh. El-Showk, M.F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, Solving the 3D Ising Model with the Conformal Bootstrap, arXiv:1203.6064 [hep-th].
- [4] F.A. Dolan and H. Osborn, Conformal Four Point Functions and the Operator Product Expansion, Nucl. Phys. B599 (2001) 459, hep-th/0011040;  
F.A. Dolan and H. Osborn, Conformal Partial Waves and the Operator Product Expansion, Nucl. Phys. B678 (2004) 491, hep-th/0309180.
- [5] M.S. Costa, J. Penedones, D. Poland and S. Rychkov, Spinning Conformal Correlators, JHEP (2011) 1111:071, arXiv:1107.3554 [hep-th];  
M.S. Costa, J. Penedones, D. Poland and S. Rychkov, Spinning Conformal Blocks, JHEP (2011) 1111:154, arXiv:1109.6321 [hep-th].
- [6] M.F. Paulos, Towards Feynman rules for Mellin amplitudes in AdS/CFT, JHEP 1110 (2011) 074, arXiv:1107.1504 [hep-th].
- [7] D. Simmons-Duffin, Projectors, Shadows and Conformal Blocks, arXiv:1204.3894 [hep-th].
- [8] F.A. Dolan and H. Osborn, Conformal Partial Waves: Further Mathematical Results, arXiv:1108.6194 [hep-th].
- [9] P. Di Francesco, P. Mathieu and D. Sénéchal, Conformal Field Theory, Springer (1996).
- [10] S. Ferrara, R. Gatto and A. F. Grillo, Properties of Partial Wave Amplitudes in Conformal Invariant Field Theories, Nuovo Cim. A 26 (1975) 226.
- [11] Al. B. Zamolodchikov, Conformal Symmetry in Two Dimensions: An Explicit Recurrence Formula for the Conformal Partial Wave Amplitude, Comm. Math. Phys. 96 (1984) 419;  
Al. B. Zamolodchikov, Conformal symmetry in two-dimensional space: Recursion representation of conformal block, Theoretical and Mathematical Physics 73 (1987) 1088;  
V. A. Belavin, N=1 Supersymmetric Conformal Block Recursion Relations, Theoretical and Mathematical Physics 152(3): 12751285 (2007);  
D. Harlow, J. Maltz and E. Witten, Analytic Continuation of Liouville Theory, JHEP 1112 (2011) 071, [arXiv:1108.4417 [hep-th]];



- V. Fateev and S. Ribault, The large central charge limit of conformal blocks, JHEP 1202 (2012) 001, arXiv:1109.6764 [hep-th].
- [12] I. Heemskerk and J. Sully, More Holography from Conformal Field Theory, JHEP 1009:099 (2010), arXiv:1006.0976 [hep-th].
- [13] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory, Nucl. Phys. B241 (1984) 333.
- [14] M.P. Mattis, Correlations In Two-dimensional Critical Theories, Nucl. Phys. B285 (1987) 671.
- [15] G. Arutyunov and S. Frolov, Three point Green function of the stress energy tensor in the AdS/CFT correspondence, Phys. Rev. D 60 (1999) 026004, [hep-th/9901121].
- [16] S. Raju, Four Point Functions of the Stress Tensor and Conserved Currents in  $AdS_4/CFT_3$ , arXiv:1201.6452 [hep-th].